EXACT FORMULATION OF ISING MODEL TRANSITIONS BETWEEN SIX MAGNETIC PHASES

by
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Model: ADama Cellular Automaton

Built upon: PyCX 0.3 Realtime Visualization Template
PyCX 0.3 Realtime Visualization Template: Written by Chun Wong
Revised by Hiroki Sayama

Requires PyCX simulator to run

PyCX available from: https://sourceforge.net/projects/pycx/

Model: ADama Checkerboard

Hypothetical optimal car formation in a 3x3 grid – «zipper merge»
A cellular automaton is a search function around a cell. The update rule is based on the neighborhood. A well known attractive-repulsive force function is the Lennard-Jones potential. We will explore if uniaxial attractive-repulsive forces can be simulated with cellular automata.
Why Uniaxial Attraction-Repulsion?

A tailing car is attracted to a frontal car’s position, but is forced to leave a trailing distance, a repulsion.

We suggest that this attraction-repulsion is a dipole in the driving direction, and in the normal direction, there is a non-dipole attraction between the vehicles which manifests as lane merging.
1st Rule: Inverse Cell Update

To achieve driving dipole, we have reversed the update of a cell. Cell’s state dictates its neighborhood.
Moore vs Von Neumann Neighborhoods
2nd Rule: Moore to Von Neumann Shift

This shift is required to simulate renormalization by decimation. We will verify this hypothesis later on.
3rd Rule: Neighborhood Configuration

Based on cell’s value, neighborhood states are tuned within range. Ideal neighborhood will turn out to be 6.

```
g = number_of_Moore_neighbors(x, y) #CA tuning
if c[x, y] == 0:
    nc[x, y] = 0 if g <= 6 else 1
    array0.append(c[x, y])

h = number_of_Neumann_neighbors(x, y) #CA tuning
if h >= 1:
    nc[x, y] = 1 if g <= 6 else 0
```
Neighbor Tuning for Ising Criticality

For 4 and 5 neighbors, has **Inverse Ising critical temperature** as the highest state 1 cell count.

For 6 neighbors however, there is another maximum susceptibility, corresponding to ferromagnetism.
4th Rule: Coupled Cellular Automata

\[ \rho(t + 1) = (1 - p) \varphi(\rho(t)) \]

\[ p(1-p): \text{Logistic map} \]

1/8: Moore Neighborhood Average

\[ p(1 - p) = \frac{1}{8} \]

\[ p^2 - p + \frac{1}{8} = 0 \]

\[ p = \frac{1}{2} \pm \frac{1}{2\sqrt{2}} \]

Left Top: Stochastic coupling mechanism evolution equation.²

Coupling Function – Magnetizing Automaton

**CRITICAL ATTRACTOR AND ISING PHASE TRANSITION**

2D Ising square lattice model’s critical inverse temperature is the **paramagnetic response**. Upper coupling is slightly striped, which is weak ferromagnetism (**ferrimagnetism**). Lower coupling is vortex shaped, which is **antiferromagnetic** behavior.

Right Figure: [http://bh.knu.ac.kr/~leehi/index.files/MPMS_HIL.pdf](http://bh.knu.ac.kr/~leehi/index.files/MPMS_HIL.pdf)
Evolution of Coupled CAs

$p(1-p)$: Logistic map; $1/8$: Moore average

\[ p(1 - p) = \frac{1}{8} \]
\[ p^2 - p + \frac{1}{8} = 0 \]
\[ p = \frac{1}{2} \pm \frac{1}{2\sqrt{2}} \]

**CLOCKWISE:** Neighbor update without coupling; Coupling without update; $\frac{1}{2} - \frac{1}{2\sqrt{2}}$ coupling, $\frac{1}{2} + \frac{1}{2\sqrt{2}}$ coupling

**RIGHT TOP:** Cell count of coupled states and uncoupled states.

**RIGHT BOTTOM:** Uncoupled states.
Trigonometric Expression of Coupling

\[ \cos^2\left(\frac{\pi}{8}\right) = \frac{1}{2} + \frac{1}{2\sqrt{2}} \]

\[ \sin^2\left(\frac{\pi}{8}\right) = \frac{1}{2} - \frac{1}{2\sqrt{2}} \]

\[ \cos^2\left(\frac{\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right) = \frac{1}{\sqrt{2}} = 2 \sin\left(\frac{\pi}{8}\right) \cos\left(\frac{\pi}{8}\right) \]

**General Expression:**

\[ a \sin^2 x - b \cos^2 x = \sin x \cos x \]
Differential Expression of Coupling

\[ a \sin^2 x - b \cos^2 x = \sin x \cos x \]

\[ \frac{\partial a(b, x)}{\partial b} = \cot^2 x; \quad \frac{\partial b(a, x)}{\partial a} = \tan^2 x \]
5th Rule: 1st Transformation

\[ a \sin^2 x - b \cos^2 x = \sin x \cos x \]
when above equation is solved:

\[ \therefore b \cot^2 x + \cot x - a = 0 \]
when the transformations are applied:

\[ -\Delta \frac{j}{i} \times \frac{i^2 \times \text{count0}^2}{64 \times \text{count1}^4} - \frac{i \times \text{count0}}{8 \times \text{count1}^2} + \left( \frac{\Delta \text{count0}}{\Delta \text{count1}} \right)_1 - \left( \frac{\Delta \text{count0}}{\Delta \text{count1}} \right)_2 \]

\[ \text{count0} = \text{cell with state 0 count} \]
\[ \text{count1} = \text{cell with state 1 count} \]
\[ j = \text{neighbor count, } i = \text{total grid} \]
\[ \sin x = 8 \text{count1} \]
\[ \cos x = \frac{\text{count0}}{\text{count1}} \]
Cotangent Graph Output of CA

Coupled and uncoupled values are arms of the cotangent graph.
Cotangent & Response Graph Intersection

Intersection points at: \( \frac{\ln(1+\sqrt{2})}{2} - \tan^2(\pi/8) \) and \( 2 \tan(\pi/8) \)
Ferrimagnetic-Ferromagnetic Transition

Lennard-Jones potential (*first-order phase transition – top, bold*)
Ferromagnetic phase (*second-order phase transition*)

Antiferromagnetic phase
(*first-order phase transition - bottom*)

\[ p = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.8535; 0.8536 \]
\[ p = \frac{1}{2} - \frac{1}{2\sqrt{2}} \approx 0.1466; 0.1465 \]
Magnetic Susceptibility v. Temperature

The transition from antiferromagnetic to ferromagnetic phase is very similar to molar magnetic susceptibility to temperature response.

\[ \chi T \propto T \]

\[ \frac{1}{T_c} \]

\[ \chi_m T \propto \text{cm}^3 K/\text{mol} \]
Second-Order Phase Transition

Inverse critical temperature and inverse magnetic susceptibility

\[ \frac{1}{\chi T} \nu \frac{1}{T} \]

1/\(\chi T\) in 1/T

\[ \frac{1}{T_c}, \frac{1}{T} \]

SECOND-ORDER PHASE TRANSITION
Symmetry axes of the Magnetic Phases

There are two symmetry axes that correspond to different phases.  

**FIRST-ORDER PHASE TRANSITION**

**SECOND-ORDER PHASE TRANSITION**

Figure: Catenoid around $p = 1/2$

Figure: Pseudosphere around $p = \ln(1+\sqrt{2})/2$

3. https://www.researchgate.net/publication/348521157_A_2D_Ising_Model_Cellular_Automaton_Mapped_Onto_Catenary_Involute
Ferrimagnetic-Ferromagnetic Transition

The transition point is when the lower bound intersection of cotangent graph and count1 graph is added upon the $\frac{1}{2}$ symmetry.

$$p_{ff} = \frac{1}{2} + \frac{\ln(1 + \sqrt{2})}{2} - \tan^2 \left( \frac{\pi}{8} \right)$$
Maximum Count1: Evasion Curve

Figure Left. Wind-blown plane problem, longest flight distance under constant wind.

Longest distance is \( \approx 1.6478 \) for \( a = 1 \).

Figure Right. Evasion curve, adapted from the wind-blown plane problem for longest distance away from the inverse Curie temperature.

Highest magnetic susceptibility = \( X_{\text{MAX}} = -\frac{1}{2} + \frac{\ln(1+\sqrt{2})}{2} + \frac{\sqrt{2}}{2} \)

\[
X_{\text{MAX}} = \frac{\ln(1+\sqrt{2})}{2} + \frac{\sqrt{2} - 1}{2} = \frac{\ln(1+\sqrt{2})}{2} + \frac{\tan\pi/8}{2}
\]


\[
\frac{1}{2} + \frac{\ln(1 + \sqrt{2})}{2} - \tan^2 \left( \frac{\pi}{8} \right) - \frac{\ln(1 + \sqrt{2})}{2} - \frac{\tan \frac{\pi}{8}}{2}
\]

which is simplified to:

\[
-\frac{1}{2} \left( 2 \cot^2 \left( \frac{3\pi}{8} \right) + \cot \frac{3\pi}{8} - 1 \right)
\]

This is the cotangent equation we have derived before:

\[ \therefore b \cot^2 x + \cot x - a = 0 \]

Its root is \( x = \tan^{-1} \left( \frac{1}{2} \right) \approx 0.46365 \)
Renormalization Recursion

0.46365 \approx 0.4656665

Recursion equation:
\cos(2 \tan^{-1} 0.4656665)

When ran sequentially, the recursion returns H, H_c and H* magnetic fields. This affirms our hypothesis.
2nd Transformation

count1 = \cos y

count0 = \sin y

When applied to:

$$-\Delta \frac{j}{i} \times \frac{i^2 \times count0^2}{64 \times count1^4} - \frac{i \times count0}{8 \times count1^2} + \left(\frac{\Delta count0}{\Delta count1}\right)_1 - \left(\frac{\Delta count0}{\Delta count1}\right)_2$$

simplifies to:

$$\frac{1}{\cos^2 y} \left(\frac{1}{8} \tan y \sec y - 1\right) = 0$$
Deriving Cotangent Outputs of CA

\[
\frac{1}{8} \tan y \sec y - 1 \approx 0.5050 \quad \text{is the } p = \frac{1}{2} \pm \frac{1}{2\sqrt{2}} \quad \text{output.}
\]

\[
\frac{1}{\cos^2 y} \left(\frac{1}{8} \tan y \sec y - 1\right) \approx 2.269 \approx \frac{2}{\ln(1+\sqrt{2})} \quad \text{is the } \left(p = \frac{1}{2} - \frac{1}{2\sqrt{2}}\right)_- \quad \text{output.}
\]

\[
1 \quad \text{is the } \left(p = \frac{1}{2} + \frac{1}{2\sqrt{2}}\right)_+ \quad \text{output (demagnetization).}
\]
Susceptibility – Curie Law

\[ \chi = \frac{c}{T \pm \theta} = \frac{1}{4 \pm 2\sqrt{2}} \] for ferrimagnetism and antiferromagnetism

\[ \chi = \frac{c}{T_c} = \frac{\ln(1+\sqrt{2})}{2} \] for paramagnetism

\[ \chi = \frac{c}{T_f} = \frac{\ln(1+\sqrt{2})}{2} + \frac{\tan\left(\frac{\pi}{8}\right)}{2} \] for global maximum of ferromagnetism
THANK YOU FOR LISTENING!

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https://github.com/goktu/ADama/blob/master/cellularautomata.py